

# Embedding of tenable and balanced urn scheme into continuous-time Pólya process

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**Abstract.** We study poissonized tenable and balanced urns on two colors, say white and blue. In particular, we look at the process obtained by embedding a generalized Pólya-Eggenberger urn into continuous time. We analyze the number of white and blue balls after a certain period of time. The asymptotic mixed moments of the process are calculated to characterize the asymptotic behavior of the process. We show that the limiting distributions of the scaled variables underlying the Bagchi-Pal urn are gamma. The technique we use is a partial differential equation that governs the process, coupled with the method of moments applied in a bootstrapped manner to produce asymptotic mixed moments. We also study the processes embedded of tenable urn schemes with random entries. Play-the-Winner scheme is given as an example. Similar results carry over to the process, with minor modifications in the methods of proof, done *mutatis mutandis*.

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## 1 Introduction

The theory of urn models have received increased attention and intensive research from probabilists and statisticians owing to its conceptual simplicity and versatility. In modern times, urn models have been recognized as a fundamental and powerful mathematical tool, and gained rising popularity among applied scientists. The applications of urn models span in a wide

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range of areas, such as algorithmics, electrical engineering, physics, sociology and biological science. The first appearance of urn models seems to be in [6], in which the author uses an urn with pebbles as an example to illustrate an empirical method of determining the number of cases. Perhaps the earliest contributions of urn models in our time are the *Pólya-Eggenberger urn* [10], which is originally used to study the spread of contagious diseases, and the *Ehrenfest urn* [11], which is proposed to model gas diffusion. We refer the readers to the texts [14, 15] for the history and more applications of urn models.

In this paper, we focus on Pólya-type urns, which are a class of urn models that involve only one urn and general methods of sampling (with replacement) under some tenable conditions that will be introduced in the sequel. A *two-color Pólya urn* is an urn containing balls of two different colors, say white and blue. The urn initially contains a certain number of balls, and grows according to certain rules. After each point of time, we draw a ball from the urn at random, observe its color and put it back to the urn. Depending on its color, a certain number of balls of each color are added. If a white ball is chosen, then we add  $a$  white balls and  $b$  blue balls to the urn; if a blue ball is chosen, we add  $c$  white balls and  $d$  blue balls to the urn. These dynamics of the urn scheme is governed by a ball *replacement matrix*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (1)$$

where the rows are indexed with white and blue from top to bottom; the columns are indexed with white and blue from the left to right. Entry  $(i, j)$  is the number of balls of color  $j \in \{\text{white, blue}\}$  added upon drawing a ball of color  $i \in \{\text{white, blue}\}$ . When entries in the replacement matrix take negative values, they refer to the number of balls removed from the urn. An urn is said to be *balanced* if the total number of balls added (regardless of the color of the ball withdrawn) is constant; that is,  $a + b = c + d = k$ , where the row sum  $k$  is called the *balance factor*. An urn is said to be *tenable* if we always can perpetuate the drawing according to replacement rules on every stochastic path, but never get “stuck.” A generalization of Pólya urns to multicolor is possible, but we only focus on two-color Pólya urns in this paper.

A *Pólya process* is a renewal process obtained by embedding a Pólya urn scheme into continuous time. Processes growing in continuous time are more realistic and applicable than those growing in discrete time. In practice, there

is no guarantee that the epochs of process are equispaced. To create a clear distinction, we adhere to the term “scheme” (i.e., Pólya urn scheme), when we speak of an urn that grows in discrete time, and to the term “process” (i.e., Pólya process), when we speak of an urn that grows in continuous time. The process is first introduced in [1], in which the authors establish a technique of embedding urn schemes into continuous-time Markov branching processes; they take *Bernard Friedman’s urn* [12] as an example. The name “Pólya process” first appears in a recent work [16].

The rest of the paper is organized as follows: In Section 2, we define Pólya processes and introduce a particular type of Pólya process that we focus on in this paper. In Section 3, we state the main result with a brief interpretation. In Section 4, we introduce the methods that we primarily rely on to prove the main theorem. This section is divided into two subsections. We give a partial differential equation that governs the process in Subsection 4.1, and establish a method with a strong flavor of bootstrapping to extract the asymptotic mixed moments of the process in Subsection 4.2. The proof of the main theorem is given in Section 5. In Section 6, we study Pólya processes obtained by embedding urn schemes with random entries into continuous time. An urn scheme with applications to clinical trials is discussed as an example. Lastly, we add some concluding remarks in Section 7.

## 2 Pólya process

We shall be mainly concerned with a (two-color) Pólya process obtained by embedding a two-color Pólya urn scheme into continuous time. The process evolves over time according to certain rules. Initially the urn contains a certain number of white and blue balls. Each ball in the urn is endowed with an independent clock that rings in time  $\text{Exp}(1)$  (an exponential random variable with mean 1). When the clock of a ball rings (at a renewal point (an epoch) in the Pólya process), the ball is immediately picked from the urn, its color is observed; the ball is then instantaneously placed back in the urn, and the rule (associated with Matrix (1)) is executed. We do not count the time loss of the selection of the ball and the execution of the rules. All new balls are endowed with their own independent clocks. The process progresses in this manner. By the memoryless property of exponential interarrival times, the process is reset to start at every epoch. Thus, the Pólya process is Markovian. In general, the process is an inhomogeneous jumping process,

and the jumps occur on every epoch. However, rate of the process changes owing to the number of ball additions.

In this paper, we consider the *Bagchi-Pal processes*, a class of Pólya processes obtained by embedding the *Bagchi-Pal urns* into continuous time. The Bagchi-Pal urn is a generalized Pólya-Eggenberger urn, originally used for estimating the computer memory requirements of 2-3 trees [3]. We give a quick word about the Bagchi-Pal urn here. The Bagchi-Pal urn is a balanced tenable urn, and its evolution is governed by the following replacement matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} k-b & b \\ c & k-c \end{pmatrix}, \quad (2)$$

where  $k$  is the balance factor, i.e.,  $k = a + b = c + d$ . Assume that the urn starts with  $W_0$  white balls, and  $B_0$  blue balls. Denote  $\tau_0 = W_0 + B_0$  the total number of balls in the urn initially. The following conditions<sup>3</sup> are needed to guarantee tenability of the urn and avoid degeneracy:

- (i)  $a + b = c + d = k \geq 1$ .
- (ii)  $W_0 + B_0 = \tau_0 \geq 1$ .
- (iii)  $a \neq c$ .
- (iv)  $b > 0$  and  $c > 0$ .
- (v) If  $a < 0$ , then  $a$  divides  $c$  and  $a$  divides  $W_0$ ; if  $d < 0$ , then  $d$  divides  $b$  and  $d$  divides  $B_0$ .

Let  $W(t)$  and  $B(t)$  respectively be the number of white and blue balls in the urn at time  $t \geq 0$  in a Bagchi-Pal process. So, at  $t = 0$ , we have  $W(0) = W_0$  and  $B(0) = B_0$ . In what follows, we denote  $\tau(t) = W(t) + B(t)$  the total number of balls in the urn at time  $t$ . The existence of the asymptotic distribution of the process  $(W(t), B(t))^\top$  has been proved in [13, Theorem 3.1]; the theorem, coupled with some classic results of branching processes in [2], yield

$$e^{-kt} \begin{pmatrix} W(t) \\ B(t) \end{pmatrix} \xrightarrow{\text{a.s.}} \text{Gamma} \left( \frac{\tau_0}{\lambda}, \lambda \right) \begin{pmatrix} v \\ 1-v \end{pmatrix}.$$

where  $\lambda$  is the principal eigenvalue of the replacement matrix,<sup>4</sup> and  $(v, 1-v)^\top$  is the corresponding normalized eigenvector.

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<sup>3</sup>See [3] for the discussion of the tenable conditions for the Bagchi-Pal urn.

<sup>4</sup>Particularly, the principal eigenvalue  $\lambda$  of Matrix (2) is equal to  $k$ .

The last display gives the asymptotic marginal distributions of  $W(t)$  and  $B(t)$  (after properly scaled). However, information about the joint limiting distribution of the process is lacking. In this paper, we establish a fundamental method to calculate all the asymptotic mixed moments of  $W(t)$  and  $B(t)$  to characterize the asymptotic behavior of the process. We also use the asymptotic mixed moments to determine the marginal limiting distributions of  $W(t)$  and  $B(t)$  (after properly scaled),<sup>5</sup> as well as the correlation between  $W(t)$  and  $B(t)$  (after properly scaled). In addition, we apply our methods to study tenable and balanced Pólya processes with replacement matrix of random entries. An embedded process of Play-the-Winner scheme is given as an example.

Before we state our main result, we introduce the mathematical notations used in this paper. The combinatorial notation  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  refers to *Stirling numbers of the second kind*, which counts the number of ways to partition a set of  $n$  objects into  $k$  nonempty subsets. The notation  $\langle x \rangle_s$  denotes the *Pochhammer symbol of the rising factorial*, defined as

$$\langle x \rangle_s = x(x+1) \cdots (x+s-1),$$

for  $x \in \mathbb{R}$  and  $s \in \mathbb{Z}^+ \cup \{0\}$ , with an interpretation of  $\langle x \rangle_0 = 1$ . The *Big O notation* defines a relation between two real-valued functions  $f(x)$  and  $g(x)$ . We have  $f(x) = O(g(x))$  if and only if there exists a positive real number  $M$  and a real number  $x_0$  such that  $|f(x)| \leq M|g(x)|$  for all  $x \geq x_0$ .

### 3 Main result

We present our main theorem in this section. The asymptotic mixed moments of  $W(t)$  and  $B(t)$  are determined explicitly. The number of white and blue balls both need the scale  $e^{kt}$  to converge non-trivially.

**Theorem 3.1.** *Let  $W(t)$  and  $B(t)$  be the number of white, respectively blue, balls in the urn at time  $t$  in a Bagchi-Pal process with replacement matrix*

$$\begin{pmatrix} k-b & b \\ c & k-c \end{pmatrix},$$

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<sup>5</sup>We achieve the same result as in [13, Theorem 3.1]

under tenable conditions (i)–(v). Assuming that the process starts with  $\tau_0 > 0$  balls, we have asymptotic mixed joint moments

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[W^i(t)B^j(t)]}{e^{k(i+j)t}} = \frac{b^j c^i}{(b+c)^{i+j}} k^{i+j} \left\langle \frac{\tau_0}{k} \right\rangle_{i+j}.$$

Accordingly, we get

$$e^{-kt} \begin{pmatrix} W(t) \\ B(t) \end{pmatrix} \xrightarrow{\mathcal{L}} \text{Gamma} \left( \frac{\tau_0}{k}, k \right) \begin{pmatrix} \frac{c}{b+c} \\ \frac{b}{b+c} \end{pmatrix},$$

and the asymptotic Pearson's correlation coefficient between  $W(t)/e^{kt}$  and  $B(t)/e^{kt}$  is 1.

We remark that the asymptotic mixed moments of  $W(t)$  and  $B(t)$  depend on the initial total number of balls ( $\tau_0$ ) in the urn, but not specifically on the initial number of white balls ( $W_0$ ), nor on the initial number of blue balls ( $B_0$ ).

## 4 Preliminaries

In this section, we introduce the methods that are very helpful in proving the main theorem stated in Section 3. The section is divided into two subsections. In Subsection 4.1, we give a partial differential equation (PDE) that governs all tenable Pólya processes. In Subsection 4.2, we introduce a method in a bootstrapped manner to determine the mixed moments of the process. This method is useful when the solution to the PDE given in Subsection 4.1 is unwieldy, which is exactly the case for the Bagchi-Pal process.

### 4.1 Partial differential equation

We consider the joint moment generating function,  $\phi(t, u, v)$ , of the process  $(W(t), B(t))^\top$ , defined as follows:

$$\phi(t, u, v) := \mathbb{E} \left[ e^{uW(t) + vB(t)} \right].$$

For general tenable Pólya urns (not necessary to be balanced) with replacement matrix (1), it has been proven that the joint process is governed by the PDE:

$$\frac{\partial \phi(t, u, v)}{\partial t} + (1 - e^{au+bv}) \frac{\partial \phi(t, u, v)}{\partial u} + (1 - e^{cu+dv}) \frac{\partial \phi(t, u, v)}{\partial v} = 0; \quad (3)$$

in [4, Lemma 2.1]. While PDE (3) is of first order, and it is known to have a general solution, the solution is given as an integration along characteristic curves. Unfortunately, these characteristics are difficult to determine in most cases. Even though we are able to determine the characteristics in few cases, integration for those characteristics is hard to calculate even for the simplest case: the Pólya process with replacement matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The solution to the associated PDE involves an integration of *Lambert W function*, which itself has an implicit definition. To the best of our knowledge, PDE (3) has only been solved in a very limited number of cases, as for instance the case of forward and backward diagonal processes [4], the Ehrenfest processes [7] and zero-balanced processes with replacement matrix of Bernoulli entries [16].

For the Bagchi-Pal process associated with replacement matrix (2), the PDE that governs the process is

$$\frac{\partial \phi}{\partial t} + (1 - e^{(k-b)u+bv}) \frac{\partial \phi}{\partial u} + (1 - e^{cu+(k-c)v}) \frac{\partial \phi}{\partial v} = 0. \quad (4)$$

We are not able to get an analytical solution to this functional equation. We shall establish a method to extract the moments, in a bootstrapped way, from PDE (4).

## 4.2 Method of bootstrapped moments

In this subsection, we introduce a method to calculate all mixed moments for the process when the solution of PDE (3) is unwieldy or not in an explicit form. We consider the mixed moments of  $W(t)$  and  $B(t)$ , i.e.,  $\mathbb{E}[W^i(t)B^j(t)]$ , for all nonnegative integers  $i$  and  $j$ , excluding the degenerate case of  $i = j = 0$ . For  $t \geq 0$ , the mixed moment  $\mathbb{E}[W^i(t)B^j(t)]$  can be obtained by applying the differential operator,  $\frac{\partial^{i+j}}{\partial u^i \partial v^j}$ , to the joint moment generating function  $\phi(t, u, v)$ , then evaluating at  $u = v = 0$ ; that is,

$$\frac{\partial^{i+j}}{\partial u^i \partial v^j} \phi(t, u, v) \Big|_{u=v=0} = \mathbb{E} \left[ \frac{\partial^{i+j}}{\partial u^i \partial v^j} e^{W(t)u+B(t)v} \Big|_{u=v=0} \right] = \mathbb{E}[W^i(t)B^j(t)].$$

We focus on the Bagchi-Pal processes. Apply the differential operator  $\frac{\partial^{i+j}}{\partial u^i \partial v^j}$  to both sides of PDE (4) and evaluate at  $u = v = 0$ . Then, we get

$$\begin{aligned} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \left( \frac{\partial}{\partial t} \phi(t, u, v) \right) \Big|_{u=v=0} &+ \frac{\partial^{i+j}}{\partial u^i \partial v^j} (1 - e^{(k-b)u+bv}) \frac{\partial \phi}{\partial u} \Big|_{u=v=0} \\ &+ \frac{\partial^{i+j}}{\partial u^i \partial v^j} (1 - e^{cu+(k-c)v}) \frac{\partial \phi}{\partial v} \Big|_{u=v=0} = 0. \end{aligned} \quad (5)$$

We evaluate three terms in the left-hand side of Equation (5) one after another. The leftmost term is given by

$$\frac{\partial^{i+j}}{\partial u^i \partial v^j} \left( \frac{\partial}{\partial t} \phi(t, u, v) \right) \Big|_{u=v=0} = \frac{d}{dt} \mathbb{E}[W^i(t)B^j(t)].$$

For the middle term in the left-hand side of Equation (5), we apply the Leibiniz rule twice and obtain

$$\begin{aligned} \frac{\partial^{i+j}}{\partial u^i \partial v^j} (1 - e^{(k-b)u+bv}) \frac{\partial \phi}{\partial u} \Big|_{u=v=0} \\ = - \sum_{r=0}^{i-1} \binom{i}{r} (k-b)^{i-r} \mathbb{E}[W^{r+1}(t)B^j(t)] \\ - \sum_{r=0}^i \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} (k-b)^{i-r} b^{j-s} \mathbb{E}[W^{r+1}(t)B^s(t)]. \end{aligned}$$

Similarly, the rightmost term in the left-hand side of Equation (5) is given by

$$\begin{aligned} \frac{\partial^{i+j}}{\partial u^i \partial v^j} (1 - e^{cu+(k-c)v}) \frac{\partial \phi}{\partial v} \Big|_{u=v=0} \\ = - \sum_{r=0}^{i-1} \binom{i}{r} c^{i-r} \mathbb{E}[W^r(t)B^{j+1}(t)] \\ - \sum_{r=0}^i \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} c^{i-r} (k-c)^{j-s} \mathbb{E}[W^r(t)B^{s+1}(t)]. \end{aligned}$$

Putting all the parts together, we obtain ordinary differential equations for



all mixed moments of  $W(t)$  and  $B(t)$ :

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[W^i(t)B^j(t)] &= \sum_{r=0}^{i-1} \binom{i}{r} (k-b)^{i-r} \mathbb{E}[W^{r+1}(t)B^j(t)] \\
&+ \sum_{r=0}^i \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} (k-b)^{i-r} b^{j-s} \mathbb{E}[W^{r+1}(t)B^s(t)] \\
&+ \sum_{r=0}^{i-1} \binom{i}{r} c^{i-r} \mathbb{E}[W^r(t)B^{j+1}(t)] \\
&+ \sum_{r=0}^i \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} c^{i-r} (k-c)^{j-s} \mathbb{E}[W^r(t)B^{s+1}(t)]. \quad (6)
\end{aligned}$$

Note that the ordinary differential equation for  $\mathbb{E}[W^i(t)B^j(t)]$  (Equation (6)) to be constructed is based on all the lower mixed moments, i.e.,  $\mathbb{E}[W^r(t)B^s(t)]$ , for  $1 \leq r+s < i+j$ , and some mixed moments at the same order, i.e.,  $\mathbb{E}[W^r(t)B^s(t)]$ , for  $r+s = i+j$ . We pursue a strategy to extract the mixed moments in a bootstrapped way (i.e., following a particular order) from the ordinary differential equations. Define  $i+j$  the *total order* of a mixed moment  $\mathbb{E}[W^i(t)B^j(t)]$ . At first, we calculate the mixed moments of low total order (starting from  $i+j = 1$ ). When calculating the mixed moments of higher total order, we plug the solutions of all the mixed moments of lower total order (that have been obtained) into the ordinary differential equations, and solve all mixed moments of the same (higher) total order simultaneously. It is expected that the ordinary differential equations of the mixed moments of high total order are complicated, since the establishment of those equations requires all mixed moments of lower total order. In addition, the higher the total order is, the more equations we need to solve.<sup>6</sup> We thus conjecture that the exact solutions of the mixed moments of high total order are very complex. In practice, we find that the mixed moments of the Bagchi-Pal process of total order 2 are already too long to present in the paper. Though the exact solutions of the mixed moments are challenging to get, we can simply extract the asymptotic mixed moments from the ordinary differential equations, and then use these asymptotic mixed moments to study the asymptotic behavior of the process.

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<sup>6</sup>In general, we have a total of  $i+j+1$  ordinary differential equations to solve at the same time to obtain the solutions of the mixed moments of total order  $i+j$ .

## 5 Proof of Theorem 3.1

In this section, we provide a proof of Theorem 3.1. Let us first simplify the notation. Set

$$m_{i,j}(t) = \mathbb{E}[W^i(t)B^j(t)].$$

In principle, we are able to obtain mixed moments  $m_{i,j}(t)$  for all  $i$  and  $j$ , by solving Equation (6). However, we have noticed that the computational complexity dramatically increases with the increase of the total order. Though the exact form of  $m_{i,j}(t)$  is complex, towards asymptotics, we only need to focus on the leading terms (the terms that have the highest power) in the expression formula of  $m_{i,j}(t)$ . The following lemma proves that the leading term of  $m_{i,j}(t)$  is an exponential function whose power is a function of  $i + j$ .

**Lemma 5.1.** *Let  $W(t)$  and  $B(t)$  be the number of white, respectively blue, balls in the urn at time  $t$  in a Bagchi-Pal process with replacement matrix (2) and under tenable conditions (i)–(v). Then, we have*

$$m_{i,j}(t) = K_{i,j}e^{k(i+j)t} + O(e^{[k(i+j)-(b+c)]t}) + O(e^{k(i+j-1)t}), \quad (7)$$

where  $K_{i,j} = K_{i,j}(b, c, k, W_0, B_0) \in \mathbb{R}$  are the coefficients for the leading terms.<sup>7</sup>

*Proof.* We prove the lemma by an induction on the total order of mixed moments. At first, we rewrite Equation (6) in terms of  $m_{i,j}(t)$  and separate the mixed moments of total order  $i + j$  from those of lower total order.

$$\begin{aligned} \frac{d}{dt}m_{i,j}(t) &= \sum_{r=0}^{i-1} \binom{i}{r} (k-b)^{i-r} m_{r+1,j}(t) + \sum_{r=0}^{i-1} \binom{i}{r} c^{i-r} m_{r,j+1}(t) \\ &\quad + \sum_{r=0}^i \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} (k-b)^{i-r} b^{j-s} m_{r+1,s}(t) \\ &\quad + \sum_{r=0}^i \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} c^{i-r} (k-c)^{j-s} m_{r,s+1}(t) \\ &= jbm_{i+1,j-1}(t) + (i(k-b) + j(k-c))m_{i,j}(t) + icm_{i-1,j+1}(t) \\ &\quad + \sum_{r=0}^{i-2} \binom{i}{r} (k-b)^{i-r} m_{r+1,j}(t) + \sum_{r=0}^{i-2} \binom{i}{r} c^{i-r} m_{r,j+1}(t) \end{aligned}$$

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<sup>7</sup>For simplicity, we drop the parameters in the parentheses from the notation.

$$\begin{aligned}
& + \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} (k-b)^{i-r} b^{j-s} m_{r+1,s}(t) \\
& + \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} \binom{i}{r} \binom{j}{s} c^{i-r} (k-c)^{j-s} m_{r,s+1}(t) \\
& + \sum_{s=0}^{j-2} \binom{j}{s} b^{j-s} m_{i+1,s}(t) + \sum_{s=0}^{j-2} \binom{j}{s} (k-c)^{j-s} m_{i,s+1}(t) \quad (8)
\end{aligned}$$

Without loss of generality, we assume that  $i + j = n$ , and our induction is on  $n$ . The base of the induction is  $n = 1$ . There are two possibilities: we either have  $i = 1, j = 0$  or  $i = 0, j = 1$ . Accordingly,  $m_{1,0}(t)$  and  $m_{0,1}(t)$  refer to the first moment for  $W(t)$  and  $B(t)$ , respectively. The corresponding differential equations (obtained by respectively setting  $i = 1, j = 0$  and  $i = 0, j = 1$  in Equation (6)) are

$$\begin{aligned}
\frac{d}{dt} \mathbb{E}[W(t)] &= (k-b) \mathbb{E}[W(t)] + c \mathbb{E}[B(t)], \\
\frac{d}{dt} \mathbb{E}[B(t)] &= b \mathbb{E}[W(t)] + (k-c) \mathbb{E}[B(t)],
\end{aligned}$$

which can be represented by the following matrix differential equation

$$\frac{d}{dt} \begin{pmatrix} \mathbb{E}[W(t)] \\ \mathbb{E}[B(t)] \end{pmatrix} = \begin{pmatrix} k-b & c \\ b & k-c \end{pmatrix} \begin{pmatrix} \mathbb{E}[W(t)] \\ \mathbb{E}[B(t)] \end{pmatrix}$$

This type of differential equation has a general solution

$$\begin{pmatrix} \mathbb{E}[W(t)] \\ \mathbb{E}[B(t)] \end{pmatrix} = c_0 e^{\lambda_0 t} \mathbf{u}_0 + c_1 e^{\lambda_1 t} \mathbf{u}_1,$$

where  $\lambda_0$  and  $\lambda_1$  are the eigenvalues of the coefficient matrix,  $\mathbf{u}_0$  and  $\mathbf{u}_1$  are the corresponding eigenvectors, and  $c_0$  and  $c_1$  are constants. Hence, we get

$$\begin{aligned}
\mathbb{E}[W(t)] &= \frac{b}{b+c} \tau_0 e^{kt} + \frac{cB_0 - bW_0}{b+c} e^{(k-b-c)t}, \\
\mathbb{E}[B(t)] &= \frac{c}{b+c} \tau_0 e^{kt} - \frac{cB_0 - bW_0}{b+c} e^{(k-b-c)t}.
\end{aligned}$$

The base is verified. To prepare for the proof of the induction, we recall some concepts from linear algebra. Let  $\mathbf{M}_n(t)$  be an  $(n+1)$ -by-1 column vector

that contains all mixed moments of total order  $n$ ; that is

$$\mathbf{M}_n(t) := \begin{pmatrix} m_{n,0}(t) \\ m_{n-1,1}(t) \\ \vdots \\ m_{1,n-1}(t) \\ m_{0,n}(t) \end{pmatrix}.$$

Let  $\mathbf{H}_n(t)$  be an  $((n^2 + n - 2)/2)$ -by-1 column vector that contains all mixed moments of total order up to  $n - 1$ ; that is

$$\mathbf{H}_n(t) := \begin{pmatrix} m_{1,0}(t) \\ m_{0,1}(t) \\ \vdots \\ m_{1,n-2}(t) \\ m_{0,n-1}(t) \end{pmatrix}.$$

For  $i+j = n$ , the ordinary differential equations (in forms of Equation (8)) can be represented by the following non-homogeneous matrix differentiation equation:

$$\frac{d}{dt}\mathbf{M}_n(t) = \mathbf{A}_n\mathbf{M}_n(t) + \mathbf{B}_n\mathbf{H}_n(t), \quad (9)$$

where  $\mathbf{B}_n$  is an  $(n + 1)$ -by- $((n^2 + n - 2)/2)$  matrix free of  $t$ , and  $\mathbf{A}_n$  is an  $(n + 1)$ -by- $(n + 1)$  tridiagonal matrix, which can be expressed explicitly as follows:

$$\mathbf{A}_n = \begin{pmatrix} n\alpha & nc & 0 & \cdots & 0 & 0 \\ b & n\alpha - \beta & (n-1)c & \cdots & 0 & 0 \\ 0 & 2b & n\alpha - 2\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n\alpha - (n-1)\beta & c \\ 0 & 0 & 0 & \cdots & nb & n\alpha - n\beta \end{pmatrix}$$

with  $\alpha = k - b$  and  $\beta = c - b$ . The tridiagonal matrix  $\mathbf{A}_n$  is a member of *Leonard pairs of the Krawtchouk type*<sup>8</sup>. This type of matrix is known to

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<sup>8</sup>We refer the readers to lecture notes [17] for discussion about Leonard pairs.

have real eigenvalues  $\lambda_0 > \lambda_1 > \dots > \lambda_n$  forming an arithmetic progression. Hence, the eigenvalues can be written as  $\lambda_0, \lambda_0 - \sigma, \lambda_0 - 2\sigma, \dots, \lambda_0 - n\sigma$  for some real positive  $\sigma$ . Since the sum of all eigenvalues of a matrix is equal to its trace, we are able to compute  $\sigma$  via the following equation:

$$\sum_{i=0}^n (\lambda_0 - i\sigma) = \sum_{i=0}^n (n\alpha - i\beta).$$

Thus, we have

$$\sigma = \frac{2\lambda_0}{n} - 2k + b + c. \quad (10)$$

Next, we determine the value of  $\lambda_0$ , in what follows, all eigenvalues for  $\mathbf{A}_n$ . It is easy to check that  $\lambda_* = nk$ ,  $\lambda_{**} = nk - (n-1)(b+c)$  and  $\lambda_{***} = nk - n(b+c)$  are three eigenvalues for  $\mathbf{A}_n$ . By previous result, we know that  $\lambda_*$ ,  $\lambda_{**}$  and  $\lambda_{***}$  can be written in terms of  $\lambda_0 - s\sigma$  for some integer  $s = 0, 1, \dots, n$ . Noticing that

$$n(b+c) = \lambda_* - \lambda_{***} \leq \lambda_0 - \lambda_n = n\sigma,$$

and

$$b+c = \lambda_{**} - \lambda_{***} \geq \lambda_0 - \lambda_1 = \sigma,$$

we conclude that  $\sigma = b+c$ ; accordingly, we get  $\lambda_0 = nk$ . Furthermore, we obtain all eigenvalues for  $\mathbf{A}_n$ :

$$\lambda_s = \lambda_0 - s\sigma = nk - s(b+c),$$

for  $s = 0, 1, \dots, n$ .

We now prove the inductive step. Assume that Equation (7) holds for all mixed moments with total order  $i+j$  up to  $n-1$ . When  $i+j = n$ , the vector  $\mathbf{H}_n(t)$  in Equation (9) can be represented as

$$\mathbf{H}_n(t) = \begin{pmatrix} K_{1,0} e^{kt} + O(e^{[k-(b+c)]t}) + O(1) \\ K_{0,1} e^{kt} + O(e^{[k-(b+c)]t}) + O(1) \\ \vdots \\ K_{1,n-2} e^{(n-1)kt} + O(e^{[(n-1)k-(b+c)]t}) + O(e^{(n-2)kt}) \\ K_{0,n-1} e^{(n-1)kt} + O(e^{[(n-1)k-(b+c)]t}) + O(e^{(n-2)kt}) \end{pmatrix},$$

by assumption. Noticing that the coefficient matrix  $\mathbf{B}_n$  is independent of  $t$ , we obtain

$$\mathbf{B}_n \mathbf{H}_n(t) = \begin{pmatrix} O(e^{(n-1)kt}) \\ O(e^{(n-1)kt}) \\ \vdots \\ O(e^{(n-1)kt}) \\ O(e^{(n-1)kt}) \end{pmatrix}.$$

Equation (9) is in a class of non-homogeneous matrix differential equations, which is known to have a general solution;<sup>9</sup> that is

$$\mathbf{M}_n(t) = e^{\mathbf{A}_n t} \mathbf{M}_n(0) + \int_0^t e^{-\mathbf{A}_n(x-t)} \mathbf{B}_n \mathbf{H}_n(x) dx. \quad (11)$$

We study the two terms in the right-hand side of Equation (11) one after another. For the first term, recall that  $n+1$  eigenvalues of  $\mathbf{A}_n$  are  $\lambda_s = nk - s(b+c)$ , for  $s = 0, 1, \dots, n$ . According to the *Sylvester's formula*, we have

$$\begin{aligned} e^{\mathbf{A}_n t} \mathbf{M}_n(0) &= \left( \sum_{s=0}^n e^{\lambda_s t} \mathcal{E}_s \right) \mathbf{M}_n(0) \\ &= \begin{pmatrix} \xi_{n,0} e^{nkt} + O(e^{[nk-(b+c)]t}) \\ \xi_{n-1,1} e^{nkt} + O(e^{[nk-(b+c)]t}) \\ \vdots \\ \xi_{1,n-1} e^{nkt} + O(e^{[nk-(b+c)]t}) \\ \xi_{0,n} e^{nkt} + O(e^{[nk-(b+c)]t}) \end{pmatrix}, \end{aligned} \quad (12)$$

where  $\mathcal{E}_s$  are idempotent matrices and  $\xi_{i,n-i} = \xi_{i,n-i}(b, c, k, W_0, B_0) \in \mathbb{R}$ , for  $i = 0, 1, \dots, n$ , are the coefficients for the leading terms herein.

The second term in the right-hand side of Equation (11) also has contributions to the leading terms. It is given by

$$\int_0^t e^{-\mathbf{A}_n(x-t)} \mathbf{B}_n \mathbf{H}_n(x) dx = \int_0^t \left( \sum_{s=0}^n e^{\lambda_s(t-x)} \mathcal{E}_s \right) \begin{pmatrix} O(e^{(n-1)kx}) \\ O(e^{(n-1)kx}) \\ \vdots \\ O(e^{(n-1)kx}) \\ O(e^{(n-1)kx}) \end{pmatrix} dx$$

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<sup>9</sup>See for instance text [9] for more details.

$$\begin{aligned}
&= \sum_{s=0}^n e^{\lambda_s t} \int_0^t e^{-\lambda_s x} \mathcal{E}_s \begin{pmatrix} O(e^{(n-1)kx}) \\ O(e^{(n-1)kx}) \\ \vdots \\ O(e^{(n-1)kx}) \\ O(e^{(n-1)kx}) \end{pmatrix} dx \\
&= \sum_{s=0}^n e^{\lambda_s t} \int_0^t \begin{pmatrix} O(e^{[(n-1)k-\lambda_s]x}) \\ O(e^{[(n-1)k-\lambda_s]x}) \\ \vdots \\ O(e^{[(n-1)k-\lambda_s]x}) \\ O(e^{[(n-1)k-\lambda_s]x}) \end{pmatrix} dx \\
&= \begin{pmatrix} \pi_{n,0} e^{nkt} + O(e^{[nk-(b+c)]t}) + O(e^{(n-1)kt}) \\ \pi_{n-1,1} e^{nkt} + O(e^{[nk-(b+c)]t}) + O(e^{(n-1)kt}) \\ \vdots \\ \pi_{1,n-1} e^{nkt} + O(e^{[nk-(b+c)]t}) + O(e^{(n-1)kt}) \\ \pi_{0,n} e^{nkt} + O(e^{[nk-(b+c)]t}) + O(e^{(n-1)kt}) \end{pmatrix}, \tag{13}
\end{aligned}$$

where  $\pi_{i,n-i} = \pi(b, c, k, W_0, B_0) \in \mathbb{R}$  are the coefficients for leading terms herein.

The proof is completed by putting equations (12) and (13) together.  $\square$

The derivation of the asymptotic mixed moments for  $W(t)$  and  $B(t)$  needs the moments for the total number of balls,  $\tau(t)$ . The moments for  $\tau(t)$  are known: The exact moment generating function of  $\tau(t)$  is determined in [4], and the exact moments for  $\tau(t)$  are given in terms of Stirling numbers of the second kind; see [8]. In the next lemma, we state (without proof) the exact moments for  $\tau(t)$ , which immediately follow the result in [8, Section 5]. We remark that the following lemma is not only applicable to the Bagchi-Pal processes, but all Pólya processes associated with tenable and balanced urn schemes.

**Lemma 5.2.** *Let  $\tau(t)$  be the total number of balls in the urn at time  $t$  in a Bagchi-Pal process with replacement matrix (2) and under tenability conditions (i)–(v). Assume that the process starts with a total of  $\tau_0 > 0$  balls. For  $n \geq 1$ , the moments of  $\tau(t)$  are*

$$\mathbb{E}[\tau^n(t)] = k^n \sum_{i=1}^n (-1)^{n-i} \begin{Bmatrix} n \\ i \end{Bmatrix} \left\langle \frac{\tau_0}{k} \right\rangle_i e^{kit}. \tag{14}$$

Towards the asymptotics of  $m_{i,j}(t)$  with fixed total order  $i + j = n$ , we need to calculate the coefficients  $K_{i,j} = K_{i,n-i}$ , for  $i = 0, 1, \dots, n$ , of the leading terms as in Equation (7).

We write the mixed moments  $m_{i,j}(t)$  in Equation (8) in terms of those given in Equation (7). The recurrence of  $K_{i,n-i}$  is constructed by dividing the equation by  $e^{knt}$  and letting  $t$  go to infinity; that is,

$$(nc + ib - ic)K_{i,n-i} = (n - i)bK_{i+1,n-i-1} + icK_{i-1,n-i+1}, \quad (15)$$

with an initial condition setting at  $i = 0$ :

$$K_{1,n-1} = \frac{c}{b}K_{0,n}.$$

Solve Equation (15) with an increasing order of  $i$ , and we obtain an iterative solution to  $K_{i,n-i}$ ,

$$K_{i,n-i} = \left(\frac{c}{b}\right)^i K_{0,n}, \quad (16)$$

for all  $i = 1, 2, \dots, n$ .

Noticing that  $\mathbb{E}[\tau^n(t)] = \mathbb{E}[(W(t) + B(t))^n]$ , we apply the *binomial theorem* and obtain

$$\sum_{i=1}^n k^n (-1)^{n-i} \left\{ \begin{matrix} n \\ i \end{matrix} \right\} \left\langle \frac{\tau_0}{k} \right\rangle_i e^{kit} = \mathbb{E}[\tau^n(t)] = \sum_{i=0}^n \binom{n}{i} m_{i,n-i}(t).$$

Dividing both sides of the last display by  $e^{knt}$ , and letting  $t$  go to infinity, we have

$$\lim_{t \rightarrow \infty} \sum_{i=1}^n k^n (-1)^{n-i} \left\{ \begin{matrix} n \\ i \end{matrix} \right\} \left\langle \frac{\tau_0}{k} \right\rangle_i \frac{e^{kit}}{e^{knt}} = \lim_{t \rightarrow \infty} \sum_{i=0}^n \binom{n}{i} \frac{m_{i,n-i}(t)}{e^{knt}}.$$

By Lemma 5.1 and the iterative relation (16), the right-hand side of this equation can be written as

$$\begin{aligned} \lim_{t \rightarrow \infty} \sum_{i=0}^n \binom{n}{i} \frac{m_{i,n-i}(t)}{e^{knt}} &= \sum_{i=0}^n \binom{n}{i} \lim_{t \rightarrow \infty} \frac{m_{i,n-i}(t)}{e^{knt}} \\ &= \sum_{i=0}^n \binom{n}{i} K_{i,n-i} \\ &= \sum_{i=0}^n \binom{n}{i} \left(\frac{c}{b}\right)^i K_{0,n} \\ &= K_{0,n} \left(1 + \frac{c}{b}\right)^n. \end{aligned}$$



Consequently, we have

$$\lim_{t \rightarrow \infty} \sum_{i=1}^n k^n (-1)^{n-i} \left\{ \begin{matrix} n \\ i \end{matrix} \right\} \left\langle \frac{\tau_0}{k} \right\rangle_i \frac{e^{kit}}{e^{knt}} = k^n \left\langle \frac{\tau_0}{k} \right\rangle_n = K_{0,n} \left( 1 + \frac{c}{b} \right)^n,$$

leading to

$$K_{0,n} = \left( \frac{b}{b+c} \right)^n k^n \left\langle \frac{\tau_0}{k} \right\rangle_n.$$

It follows that

$$K_{i,n-i} = \left( \frac{c}{b} \right)^i \left( \frac{b}{b+c} \right)^n k^n \left\langle \frac{\tau_0}{k} \right\rangle_n = \frac{b^{n-i} c^i}{(b+c)^n} k^n \left\langle \frac{\tau_0}{k} \right\rangle_n,$$

for all  $i = 0, 1, 2, \dots, n$  and  $n \geq 1$ . Replace  $n-i$  by  $j$  in the last display, and we get

$$\lim_{t \rightarrow \infty} \frac{m_{i,j}(t)}{e^{k(i+j)t}} = K_{i,j} = \frac{b^j c^i}{(b+c)^{i+j}} k^{i+j} \left\langle \frac{\tau_0}{k} \right\rangle_{i+j}. \quad (17)$$

Set  $j = 0$ , and we can rewrite Equation (17) as

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[ \left( \frac{W(t)}{e^{kt}} \right)^i \right] = \left( \frac{ck}{b+c} \right)^i \left\langle \frac{\tau_0}{k} \right\rangle_i,$$

where the right-hand side of the display is the  $i$ -th moment of gamma random variable with parameters  $\tau_0/k$  and  $ck/(b+c)$ .

Similarly, setting  $i = 0$ , we get

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[ \left( \frac{B(t)}{e^{kt}} \right)^j \right] = \left( \frac{bk}{b+c} \right)^j \left\langle \frac{\tau_0}{k} \right\rangle_j,$$

where the right-hand side of the display is the  $j$ -th moment of gamma random variable with parameters  $\tau_0/k$  and  $bk/(b+c)$ .

Therefore, we conclude that

$$e^{-kt} \begin{pmatrix} W(t) \\ B(t) \end{pmatrix} \xrightarrow{\mathcal{L}} \text{Gamma} \left( \frac{\tau_0}{k}, k \right) \begin{pmatrix} \frac{c}{b+c} \\ \frac{b}{b+c} \end{pmatrix},$$

Lastly, we calculate the Pearson's correlation coefficient between  $W(t)/e^{kt}$  and  $B(t)/e^{kt}$ , as  $t \rightarrow \infty$ . Set  $i = j = 1$ ,  $i = 1, j = 0$  and  $i = 0, j = 1$  in

Equation (17), respectively, and we get

$$\begin{aligned}\lim_{t \rightarrow \infty} \text{Cov} \left( \frac{W(t)}{e^{kt}}, \frac{B(t)}{e^{kt}} \right) &= \lim_{t \rightarrow \infty} \left( \mathbb{E} \left[ \frac{W(t)}{e^{kt}} \cdot \frac{B(t)}{e^{kt}} \right] - \mathbb{E} \left[ \frac{W(t)}{e^{kt}} \right] \mathbb{E} \left[ \frac{B(t)}{e^{kt}} \right] \right) \\ &= \frac{bck\tau_0}{(b+c)^2}.\end{aligned}$$

Set  $i = 2, j = 0$  and  $i = 0, j = 2$ , and we respectively get the asymptotic variances for  $W(t)/e^{kt}$  and  $B(t)/e^{kt}$ :

$$\begin{aligned}\lim_{t \rightarrow \infty} \text{Var} \left( \frac{W(t)}{e^{kt}} \right) &= \frac{c^2 k \tau_0}{(b+c)^2} \\ \lim_{t \rightarrow \infty} \text{Var} \left( \frac{B(t)}{e^{kt}} \right) &= \frac{b^2 k \tau_0}{(b+c)^2}\end{aligned}$$

Hence, the asymptotic Pearson's correlation coefficient between  $W(t)/e^{kt}$  and  $B(t)/e^{kt}$  is

$$\lim_{t \rightarrow \infty} \text{Corr} \left( \frac{W(t)}{e^{kt}}, \frac{B(t)}{e^{kt}} \right) = \frac{bck\tau_0/(b+c)^2}{\sqrt{c^2 k \tau_0/(b+c)^2} \sqrt{b^2 k \tau_0/(b+c)^2}} = 1.$$

## 6 An application to balanced Pólya processes with replacement matrix of random entries

In this section, we present an application of our method to tenable and balanced processes with replacement matrix of random entries. We take the process embedded of Play-the-Winner scheme as an example. Similar results of Theorem 3.1 carry over to this process with minor modifications to the methods of proof. So, we will just state the key points in the proof, but omit those analogous arguments.

*Play-the-Winner* scheme is an adaptive randomization scheme in clinical trials. Suppose that there are two treatments (treatment 1 and treatment 2), and a clinician selects a treatment for the next patient according to the following rules. The clinician randomly selects a ball from a two-color (white and blue) urn, observes its color, and returns it back to the urn. If the ball is white, treatment 1 is given to the patient. If the treatment (treatment 1) succeeds, a white ball is added to the urn; otherwise, a blue ball is added to the urn. On the other hand, if the ball sampled by the clinician is blue,

treatment 2 is given to the patient. If the treatment (treatment 2) succeeds, a blue ball is added to the urn; otherwise, a white ball is added to the urn. Such scheme is always in favor of the successful treatment. Suppose that the success rate for each of the treatments stays unchanged, say  $p_1$  for treatment 1 and  $p_2$  for treatment 2, the dynamics of Play-the-Winner schemes can be represented by the next replacement matrix:

$$\begin{pmatrix} \mathcal{B}_1 & 1 - \mathcal{B}_1 \\ 1 - \mathcal{B}_2 & \mathcal{B}_2 \end{pmatrix}, \quad (18)$$

where  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are Bernoulli random variables with success rates  $p_1$  and  $p_2$ , respectively. It is obvious that the urn is balanced. The urn is tenable as well, since all entries in Matrix (18) are non-negative.

The PDE that governs the Pólya processes embedded by tenable urn schemes with integer-valued random entries has been determined and is given in [15, Lemma 4.1]. Thus, the PDE for the Pólya processes associated with Play-the-Winner schemes is

$$\frac{\partial \phi}{\partial t} + [1 - (e^u(1 - p_1) + e^v p_1)] \frac{\partial \phi}{\partial u} + [1 - (e^v(1 - p_2) + e^u p_2)] \frac{\partial \phi}{\partial v} = 0. \quad (19)$$

We apply the differential operator  $\frac{\partial^{i+j}}{\partial u^i \partial v^j}$  to both sides of PDE (19), then evaluate the equation at  $u = v = 0$ , and obtain

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[W^i(t)B^j(t)] &= jp_1 \mathbb{E}[W^{i+1}(t)B^{j-1}(t)] + ip_2 \mathbb{E}[W^{i-1}(t)B^{j+1}(t)] \\ &\quad + (i(1 - p_1) + j(1 - p_2)) \mathbb{E}[W^i(t)B^j(t)] \\ &\quad + (1 - p_1) \sum_{r=0}^{i-2} \binom{i}{r} \mathbb{E}[W^{r+1}(t)B^j(t)] \\ &\quad + p_2 \sum_{r=0}^{i-2} \binom{i}{r} \mathbb{E}[W^r(t)B^{j+1}(t)] \\ &\quad + p_1 \sum_{s=0}^{j-2} \binom{j}{s} \mathbb{E}[W^{i+1}(t)B^s(t)] \\ &\quad + (1 - p_2) \sum_{s=0}^{j-2} \binom{j}{s} \mathbb{E}[W^i(t)B^{s+1}(t)]. \end{aligned}$$

We claim that

$$\mathbb{E}[W^i(t)B^j(t)] = M_{i,j}e^{(i+j)t} + O(e^{[(i+j)-(p_1+p_2)]t}) + O(e^{(i+j-1)t}),$$

where  $M_{i,j} = M_{i,j}(p_1, p_2, W_0, B_0) \in \mathbb{R}$  are the coefficients for the leading terms. We prove our claim by an induction on the total order  $i + j$  of the mixed moments. The base of the induction ( $i + j = 1$ ) is verified by respectively finding the first moment for  $W(t)$  and  $B(t)$ , which are

$$\begin{aligned}\mathbb{E}[W(t)] &= \frac{p_2}{p_1 + p_2}\tau_0 e^t + \frac{p_2 B_0 - p_1 W_0}{p_1 + p_2}e^{(1-p_1-p_2)t}, \\ \mathbb{E}[B(t)] &= \frac{p_1}{p_1 + p_2}\tau_0 e^t - \frac{p_2 B_0 - p_1 W_0}{p_1 + p_2}e^{(1-p_1-p_2)t}.\end{aligned}$$

The inductive step is analogous to the proof of Lemma 5.1.

Next, we determine the coefficients  $M_{i,j}$ . Fix  $i + j = n$ , and we construct a recurrence for  $M_{i,n-i}$ ,

$$(ip_1 + (n-i)p_2)M_{i,n-i} = (n-i)p_1 M_{i+1,n-i-1} + ip_2 M_{i-1,n-i+1},$$

with an initial condition

$$M_{1,n-1} = \frac{p_2}{p_1}M_{0,n}.$$

Solve the recurrence iteratively with an increasing order of  $i$ , and we get

$$M_{i,n-i} = \left(\frac{p_2}{p_1}\right)^i M_{0,n},$$

for all  $i = 0, 1, \dots, n$ . Since the urn associated with Play-the-Winner scheme is tenable and balanced, the moments of  $\tau_n(t)$  presented in Lemma 5.2 still hold for this case, and  $k$  therein takes value 1 particularly. Solve  $M_{0,n}$  in an analogous manner in solving  $K_{0,n}$  in Section 5, and we get

$$M_{0,n} = \frac{p_1^n}{(p_1 + p_2)^n} \langle \tau_0 \rangle_n.$$

It follows that

$$M_{i,n-i} = \frac{p_1^{n-i} p_2^i}{(p_1 + p_2)^n} \langle \tau_0 \rangle_n.$$

Replace  $n - i$  by  $j$ , and we arrive at

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[W^i(t)B^j(t)]}{e^{(i+j)t}} = \frac{p_1^j p_2^i}{(p_1 + p_2)^{i+j}} \langle \tau_0 \rangle_{i+j}.$$

We respectively set  $j = 0$  and  $i = 0$ , and find that

$$\begin{aligned}\lim_{t \rightarrow \infty} \mathbb{E} \left[ \left( \frac{W(t)}{e^t} \right)^i \right] &= \left( \frac{p_2}{p_1 + p_2} \right)^i \langle \tau_0 \rangle_i, \\ \lim_{t \rightarrow \infty} \mathbb{E} \left[ \left( \frac{B(t)}{e^t} \right)^j \right] &= \left( \frac{p_1}{p_1 + p_2} \right)^j \langle \tau_0 \rangle_j.\end{aligned}$$

Thus, we conclude that

$$e^{-t} \begin{pmatrix} W(t) \\ B(t) \end{pmatrix} \xrightarrow{\mathcal{L}} \text{Gamma}(\tau_0, 1) \begin{pmatrix} \frac{p_2}{p_1 + p_2} \\ \frac{p_1}{p_1 + p_2} \end{pmatrix}.$$

In addition, as  $t \rightarrow \infty$ , we have

$$\begin{aligned}\lim_{t \rightarrow \infty} \text{Cov} \left( \frac{W(t)}{e^t}, \frac{B(t)}{e^t} \right) &= \frac{p_1 p_2}{(p_1 + p_2)^2} \tau_0, \\ \lim_{t \rightarrow \infty} \text{Var} \left( \frac{W(t)}{e^t} \right) &= \frac{p_2^2}{(p_1 + p_2)^2} \tau_0, \\ \lim_{t \rightarrow \infty} \text{Var} \left( \frac{B(t)}{e^t} \right) &= \frac{p_1^2}{(p_1 + p_2)^2} \tau_0,\end{aligned}$$

leading to

$$\lim_{t \rightarrow \infty} \text{Corr} \left( \frac{W(t)}{e^t}, \frac{B(t)}{e^t} \right) = 1.$$

## 7 Concluding remarks

We add some concluding remarks in this section. In the discrete-time Bagchi-Pal urn schemes, the numbers of white balls  $W_n$  and blue balls  $B_n$  both asymptotically have normal distributions after properly scaled [3], under an additional condition of  $k - b - c \leq k/2$ . By contrast, in the continuous-time Bagchi-Pal processes that arise in the poissonized Bagchi-Pal urn schemes, we need to appropriately scale the corresponding random variables  $W(t)$  and  $B(t)$  to obtain limiting distributions, and both limits are gamma. We see that embedding into continuous time produces remarkably different results.

Since the asymptotic correlation coefficient between  $W(t)/e^{kt}$  and  $B(t)/e^{kt}$  in the Bagchi-Pal processes is equal to 1, the limiting random variables are

linearly related. Even though both of the limiting random variables have marginal gamma distributions, the joint limiting distribution does not fit any known types of bivariate gamma distributions in [5, Chapter 8]. In this paper, we introduce a method of moments to characterize the asymptotic behavior of the joint distribution of a large class of Pólya processes. However, the explicit formula of the joint limiting distribution of the process remains unknown for further investigation.

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